ON THE DOMINANCE
OF PARTIAL DIFFERENTIAL OPERATORS[1]

BY

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1. **Introduction.** Let G be a domain in Euclidean n-space and let V be a set of functions defined on G. If $P_1, \cdots, P_q$ are partial differential operators, a fundamental problem in many investigations has been to determine those operators R for which one has estimates of the form

$$(1.1) \qquad \| Ru \|_0 \leqq C \sum_{k=1}^{q} \| P_k u \|_k$$

holding for all $u \in V$, where the $\| \ \|_i$, $i = 0, 1, \cdots, q$, are suitable norms. In fact, one can safely say that it is a rare investigation in partial differential equations which does not require estimates of the form (1.1). Of course, the validity of (1.1) depend upon all contributing factors — G, the P_k, R, V as well as the norms. In many instances the choice of these factors is by no means clear for the particular problem at hand and must be left to the ingenuity of the investigator.

When (1.1) holds we say that R is dominated by the P_k over the set V with respect to the given norms. The general problem, given G, V, the P_k and the norms, is to determine all operators R which are so dominated. This problem has not been solved in general even for some simple situations. However, many important results are known. The most popular norms are the maximum, L^p, and Hölder norms, with or without weight functions. Usually V is a space of functions which satisfy given conditions at the boundary. When employing L^2 norms, the right-hand side can be generalized to quadratic forms. There are results for systems as well.

If V is determined by equations of the form

$$(1.2) \qquad Q_j u = 0, \qquad j = 1, 2, \cdots, q_1,$$

holding on the boundary ∂G of G, one can formulate the problem in a slightly more general framework, namely to determine all R for which

$$(1.3) \qquad \|Ru\|_0 \leq C(\sum \|P_k u\|_k + \sum ||| Q_j u |||_j)$$

holds for all functions u, where the $\||\ \||_j$ are suitable boundary norms. Of course, (1.3) will imply (1.1) for $u \in V$.

It would be extremely difficult to mention all known results for problems of this sort. Since we shall be primarily concerned with the L^2 norm we mention some recent investigations for that norm (without weight functions).

Aronszajn [3] investigated the case when G is bounded and $P_q = 1$. If m' is the maximum order of the P_k and the P_k dominate all operators of order $\leqq m'$ on V, he calls the P_k "coercive" over V. He determined necessary and sufficient conditions(2) for the P_k to be coercive over $C^{m'}(\bar{G})$. For the case $q = 2$ and P_1 a strongly elliptic operator, Guseva [7], Nirenberg [10], and Browder [4] showed that P_1 is coercive over the set of those $u \in C^{m'}(\bar{G})$ which have zero Dirichlet data on the boundary. In [4; 10] an inequality due to Gårding [6] was employed. Later, this was generalized to an arbitrary elliptic operator P_1 satisfying a "root" condition by Schechter [11]. In addition, more general spaces V defined by the equations of the form (1.2) where the Q_j are differential operators were considered by Agmon-Douglis-Nirenberg [2] and Browder [5]. The case $q > 2$ was also generalized by Agmon [1] and Schechter [12] to include the more general space V.

When G is bounded, $q = 1$, P_1 has constant coefficients, and V is the set $C_0^{m'}(G)$ of functions $u \in C^{m'}(\bar{G})$ with compact support in G, Malgrange [9] and Hörmander [8] showed that P_1 always dominates $R = 1$. When R is dominated by P_1 over $C_0^{m'}(G)$, Hörmander [8] calls it "weaker" than P_1. He gives a criterion for determining all operators R which are weaker than a given P_1(3). Thomée [14] has treated the case when $R = 1$ and V is more general. He assumes that P_1 is homogeneous and that G has a portion of its boundary situated in a hyperplane. V was taken to be the set of functions which satisfy zero Dirichlet-type data on the planar part of the boundary and which vanish near the remaining portion. When $n = 2$ and P_1 has simple (complex) characteristics, he also considered more general boundary conditions of the type (1.3). He calls the boundary condition "locally cogent" if P_1 dominates $R = 1$.

In this paper we consider G with part $\partial_1 G$ of its boundary situated in a hyperplane. We consider functions $u \in C_0^{m'}(G \cup \partial_1 G)$, i.e., function in $C^{m'}(\bar{G})$ which vanish near $\partial G - \partial_1 G$ and outside compact sets. R and the P_k are homogeneous, constant coefficient operators of orders m, m_k, respectively, $k = 1, 2, \cdots, q$. The boundary conditions are expressed in the form (1.2) where the Q_j are also homogeneous, constant coefficient operators. We employ norms slightly more general than the $L^2(G)$ norm.

Our general results are stated in §§2 and 6. As a special case they imply sufficient conditions for an inequality of the form

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(2) We do not give the regularity assumptions made for some of the reported results.

(3) His methods carry over immediately to the case $q > 1$, as long as the P_k have constant coefficients.

$$(1.4) \qquad \|Ru\| \leq C\left(\sum_k \|P_k u\| + \sum_j \||Q_j u\||_j\right)$$

to hold for all $u \in C_0^\infty(G \cup \partial_1 G)$, where $\| \ \|$ is the $L^2(G)$ norm and the $\|| \ \||_j$ are suitable boundary norms. To give an idea as to their nature, we describe one theorem for $n = 2$. It is simpler to state and indicates what one can expect for $n > 2$.

Assume that each $m_k \geq m$. We consider the characteristic polynomials $P_k(\xi, \eta)$ of the P_k, where η corresponds to the coordinate normal to the plane portion of ∂G. The multiplicities of the real η roots of $P_k(\xi, \eta)$ are decreased by $m_k - m$ and we consider the greatest common divisor $P(\xi, \eta)$ of the resulting polynomials. We let $P^+(1, \eta)$ (resp. $P^-(1, \eta)$) be the largest divisor of $P(1, \eta)$ having only roots with non-negative (resp. nonpositive) imaginary parts and denote its degree by m^+(resp. m^-). Then our theorem states that there is an inequality of the form (1.4) holding for all $u \in C_0^{m'}(G \cup \partial_1 G)$ if R is weaker than the P_k and there are m^+ (resp. m^-) of the polynomials $Q_j(1, \eta)$ which are of degree less than m' and are linearly independent modulo $P^+(1, \eta)$ (resp. $P^-(1, \eta)$). More precise statements can be found in §2.

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2. Definitions and partial results. Let E^n denote Euclidean n-space with generic points $x = (x_1, \cdots, x_n)$. Let G be a domain in the halfspace $x_n \geq 0$ with a portion $\partial_1 G$ of its boundary ∂G contained in the hyperplane $x_n = 0$. For any integer $m \geq 1$ we consider linear homogeneous partial differential operators of order m with constant coefficients. We write them in the form

$$(2.1) \qquad P = \sum_{|\mu| + s = m} a_{\mu s} D^\mu D_n^s,$$

where $\mu = (\mu_1, \cdots, \mu_{n-1})$ is an $n - 1$ dimensional multi-index of non-negative integers, $|\mu| = \mu_1 + \cdots + \mu_{n-1}$ the $a_{\mu s}$ are complex constants,

$$D^\mu = (-i)^{|\mu|} \partial^{|\mu|} / \partial x_1^{\mu_1} \cdots \partial x_{n-1}^{\mu_{n-1}},$$

and

$$D_n^s = (-i)^s \partial^s / \partial x_n^s.$$

We shall consider the class $C_0^\infty(G \cup \partial_1 G)$ of complex valued, infinitely differentiable functions defined on the closure $\bar{G}$ of G which have compact support in $G \cup \partial_1 G$, i.e., which vanish near $\partial G - \partial_1 G$ and outside compact sets. The characteristic polynomial corresponding to P is

$$(2.2) \qquad P(\xi, \eta) = \sum_{|\mu| + s = m} a_{\mu s} \xi^\mu \eta^s,$$

where $\xi = (\xi_1, \cdots, \xi_{n-1})$ is an $n - 1$ dimensional vector and

$$\xi^\mu = \xi_1^{\mu_1} \cdots \xi_{n-1}^{\mu_{n-1}}.$$

Let P_k, Q_j be homogeneous, constant coefficient operators of orders m_k, v_j, respectively, $k = 1, 2, \cdots, n_1; \; j = 1, 2, \cdots, n_2$. Let

$$(2.3) \qquad\qquad P_k(\xi, \eta) = \sum_{|\mu|+s = m_k} a_{k\mu s}\xi^\mu \eta^s, \qquad k = 1, 2, \cdots, n_1,$$

and

$$(2.4) \qquad\qquad Q_j(\xi, \eta) = \sum_{|\mu|+s = v_j} b_{k\mu s}\xi^\mu \eta^s, \qquad j = 1, 2, \cdots, n_2,$$

be the corresponding characteristic polynomials. For each ξ such that $|\xi|^2 \equiv \xi_1^2 + \cdots + \xi_{n-1}^2 = 1$, let $P_\xi(\eta)$ be the greatest common divisor of the $P_k(\xi, \eta)$, and let $P_\xi^+(\eta)$ be the largest factor of $P_\xi(\eta)$ having only roots with non-negative imaginary parts. Denote the degree of $P_\xi^+(\eta)$ by $m^+(\xi)$.

For a function $u \in C_0^\infty(G \cup \partial_1 G)$ we shall consider norms defined by means of Fourier transforms. We consider functions in $C_0^\infty(G \cup \partial_1 G)$ to be defined as zero outside $\bar{G}$ and set[4]

$$\tilde{u}(\xi, \eta) = \int e^{-i(\xi x' + \eta y)} u(x', y)dx'dy,$$

$$\hat{u}(\xi) = i \int e^{-i\xi x'} u(x', 0)dx',$$

where $x' = (x_1, \cdots, x_{n-1})$, $y = x_n$, and $\xi x' = \xi_1 x_1 + \cdots + \xi_{n-1}x_{n-1}$. For any real number r we define

$$|u|_r^2 = \int |\xi|^{2r}|\tilde{u}|^2 d\xi d\eta,$$

$$\langle u \rangle_r^2 = \int |\xi|^{2r}|\hat{u}|^2 d\xi.$$

By Parseval's identity, $|u|_0$ is equivalent to the $L^2(G)$ norm of u.

THEOREM 2.1. *Let R be a homogeneous constant coefficient operator of order m with characteristic polynomial $R(\xi, \eta)$. Then there is a constant C such that*

$$(2.5) \qquad |Ru|_r \leqq C \left(\sum_k |P_k u|_{r+m-m_k} + \sum_j \langle Q_j u \rangle_{r+m-v_j-1/2} \right)$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$ and real numbers r, if

(a) *there is an integer $m_0 \geqq 0$ such that $a_{k\mu s} = 0$ for $s > m_0$ and all k, μ, while*

$$\sum_k \left| \sum_\mu a_{k\mu m} \; \xi^\mu \right| \neq 0$$

for $|\xi| = 1$;

(b) *there is a constant K_0 such that*

$$(2.6) \qquad |R(\xi,\eta)|^2 \leqq K_0 \sum |\xi|^{2(m-m_k)} |P_k(\xi,\eta)|^2$$

for all real ξ,η, $\xi \neq 0$;

(c) *the $Q_j(\xi,\eta)$ cover the $P_k(\xi,\eta)$. By this we mean that for each ξ such that $|\xi| = 1$ there are $m^+(\xi)$ of the $Q_j(\xi,\eta)$ which are of degree less than m_0 in η and are linearly independent modulo $P_\xi^+(\eta)$.*

We have so far made no restriction on G other than that it be contained in the halfspace $x_n \geqq 0$. We now assume that it is bounded. Our general results for this case (Theorems 6.1 and 6.2) are too complicated to state here. We content ourselves with considering special cases.

Let $P(\xi,\eta)$ be the characteristic polynomial corresponding to a constant coefficient partial differential operator P. At each point ξ of $|\xi| = 1$ we can factor $P(\xi,\eta)$ in the form

$$P(\xi,\eta) = a(\xi) \prod_{t=0}^{N(\xi)} (\eta - \lambda_t(\xi))^{s_t(\xi)},$$

where $a(\xi) \neq 0$, $N(\xi)$, and $s_t(\xi)$ are, in general, discontinuous functions of ξ. The $\lambda_t(\xi)$ are considered roots of $P(\xi,\eta)$ with multiplicities $s_t(\xi)$, respectively. For any integer q we set

$$s_t^q(\xi_0) = \max\left[s_t(\xi_0) - q, 0\right]$$

if $\lambda_t(\xi_0)$ is real and $\mathrm{I\,m}\,\lambda_t(\xi) \leqq 0$ in a neighborhood of ξ_0. Otherwise we set

$$s_t^q(\xi_0) = s_t(\xi_0).$$

We define

$$P^{(q)}(\xi,\eta) \equiv a(\xi) \prod (\eta - \lambda_t(\xi)^{s_t^q(\xi)}$$

when $q \geqq 0$, and $P^{(q)}(\xi,\eta) = 0$ otherwise. We say that the polynomials $P_k(\xi,\eta)$ are *collectively normal* if they satisfy hypothesis (a) of Theorem 2.1.

THEOREM 2.2. *For any set of non-negative integers ρ, $\{l_k\}_{k=1}^{n_1}$, satisfying $\rho \leqq \max l_k$ and any real $r \geqq \max_{j,k}(m_k, v_j + 1)$ there is a constant C such that*

$$(2.7) \qquad |Ru|_{r-m} \leqq C\left\{ \sum_k |P_k u|_{r-m_k+l_k} + \sum_j \langle Q_j u \rangle_{r-v_j+\rho} \right\}$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$, if
  (a) *the polynomials $P_k(\xi,\eta)$ are collectively normal;*
  (b) *there are non-negative integers $\rho_k \leqq \min(\rho + 1, l_k)$ such that*

$$(2.8) \qquad |R(\xi,\eta)|^2 \leqq K \sum_k \sum_{s=0}^{\rho_k} |\xi|^{2(m-m_k+s)} \left| \frac{\partial^s P_k(\xi,\eta)}{\partial\eta^s} \right|^2$$

for all real $\xi, \eta, \xi \neq 0$;

   (c)   *the $Q_j(\xi,\eta)$ cover the polynomials $P_k^{(\sigma_k)}(\xi,\eta)$, where*

$$\sigma_k = \min(l_k - \rho_k, l_k - \rho).$$

THEOREM 2.3. *Inequality (2.7) holds for any operator R of order m if hypotheses (a)–(c) of Theorem 2.2 are replaced by the following:*

   (a′)   *the polynomials $P_k(\xi,\eta)$ are collectively normal and $m_0 \geqq m$;*

   (b′)   *for no ξ of $|\xi| = 1$ does $P_\xi(\eta)$ have any real roots with multiplicities greater than $\min(\rho + 1, \min l_k)$;*

   (c′)   *the $Q_j(\xi,\eta)$ cover the polynomials $P_k^{(l_k-\rho)}(\xi,\eta)$.*

We end this section by stating some theorems in two dimensions. They are slightly stronger than Theorems 2.2 and 2.3 and indicate the power of our general results (Theorems 6.1 and 6.2). We now assume that $n = 2$. Hence $\xi = \xi_1$, a number. The only restriction we make on G is that it be contained in a strip $0 \leqq x_n \leqq d$.

THEOREM 2.4. *For any set of non-negative numbers $\{l_k\}$ there is a constant C such that*

$$(2.9) \qquad |Ru|_{r-m} \leqq C \left\{ \sum_k |P_k u|_{r-m_k+l_k} + \sum_j \sum_{s=0}^{1} \langle Q_j u \rangle_{r-v_j+((s-1)/2)} \right\}$$

for all real r and $u \in C_0^\infty(G \cup \partial_1 G)$, if

   (a)   *there is a constant K_1 such that*

$$(2.10) \qquad |R(\xi,\eta)|^2 \leqq K_1 \sum_k |\xi|^{2(m-m_k+l_k)} \sum_{s+t=l_k} \left| \frac{\partial^{s+t} P_k(\xi,\eta)}{\partial\xi^s \partial\eta^t} \right|^2$$

for all real $\xi, \eta, \xi \neq 0$;

   (b)   *the $Q_j(\xi,\eta)$ cover the polynomials $P_k^{(l_k)}(\xi,\eta)$.*

A special case of Theorem 2.4 can be put in a convenient form if we employ the following concept due to Hörmander [8]. The polynomial $R(\xi,\eta)$ is said to be *weaker* than the polynomials $P_k(\Sigma,\eta)$ if there is a constant C such that

$$(2.11) \qquad |R(\xi,\eta)|^2 \leqq C \sum \left| \frac{\partial^{s+t} P_k(\xi,\eta)}{\partial\xi^s \partial\eta^t} \right|^2,$$

for all real ξ, η, where summation is taken over all non-negative integers s and t as well as all polynomials $P_k(\xi,\eta)$. By Hörmander's result [8], this is a necessary and sufficient condition for

$$|R\phi|_0 \leqq C \sum |P_k\phi|_0$$

to hold for $\phi \in C_0^\infty(G)$. When each $m_k \geqq m$ we have

THEOREM 2.5. *There is a constant C such that*

$$|Ru|_\tau \leq C\left(\sum_k |P_k u|_\tau + \sum_j \sum_{s=0}^1 \langle Q_j u \rangle_{\tau+m-\nu_j+((s-1)/2)} \right)$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$ *and real* τ *if*
   (a) $R(\xi, \eta)$ *is weaker than the* $P_k(\xi, \eta)$;
   (b) *the* $Q_j(\xi, \eta)$ *cover the* $P_k^{(m_k-m)}(\xi, \eta)$.

The proof of Theorem 2.1 will be given in §4. It is based on the results of §3. The proofs of Theorems 2.2–2.5 will be given in §6 together with more general statements and theorems. The method of proof is developed in §5.

3. Unbounded domains in halfspace. In this section we allow G to be unbounded. For any integer $m \geq 1$ we consider polynomials in η of the form

$$P(\xi, \eta) = \sum_{s=0}^m a_s(\xi)\eta^s$$

where each coefficient $a_s(\xi)$ is an analytic function of ξ on some closed subset Ξ of the unit sphere $|\xi| = 1$ and is homogeneous in ξ of degree $m - s$. Such a polynomial will be called a *characteristic polynomial* of order m. Its *normal order* is the highest index h for which $a_h(\xi) \not\equiv 0$ in Ξ. Thus any characteristic polynomial $P(\xi, \eta)$ can be written in the form

$$P(\xi, \eta) = a_h(\xi) \prod_{t=1}^h (\eta - \lambda_t(\xi)),$$

where each root $\lambda_t(\xi)$ is homogeneous of degree one. Moreover, it is well known that when $a_h(\xi) \neq 0$, Ξ can be decomposed into a finite number of sets on each of which the $\lambda_t(\xi)$ are continuous, single valued functions (cf., e.g., [15, p. 126]).

As before, we consider a function $u \in C_0^\infty(G \cup \partial_1 G)$ to be defined as a zero outside $\bar{G}$ and let

$$\tilde{u}(\xi, \eta) = \int e^{-i(\xi x' + \eta y)} u(x', y) dx' dy$$

denote its Fourier transform, where $x' = (x_1, x_2, \cdots, x_{n-1})$, $y = x_n$, and $\xi x' = \xi_1 x_1 + \cdots + \xi_{n-1} x_{n-1}$. We also consider the "tangential" Fourier transform

$$\hat{u}(\xi) = i \int e^{-i\xi x'} u(x', 0) dx'$$

and set

$$\omega_s = |\xi|^{-s-1} (D_n^s u)\hat{\ } ; \quad \Omega_s = \sum_{t=0}^{s-1} \eta^{s-t-1} (D_n^t u)\hat{\ } , \quad s = 0, 1, 2, \cdots,$$

$$lu = (\hat{u}, (D_n u)\hat{\ } , \cdots, (D_n^j u)\hat{\ } , \cdots),$$

$$Lu = (\Omega_0, \Omega_1, \cdots, \Omega_j, \cdots).$$

Corresponding to each characteristic polynomial $P(\xi, \eta)$ we define the linear operators

$$(Pu)^\sim = P(\xi, \eta)\tilde{u} + P(\xi, Lu),$$

$$(Pu)^\wedge = P(\xi, lu)$$

where, for any vector $w = (w_0, w_1, \cdots, w_s, \cdots)$ we employ the notation

$$P(\xi, w) = \sum_{s=0}^{m} a_s(\xi)w_s.$$

Let $R(\xi, \eta)$, $P_k(\xi, \eta)$, $Q_j(\xi, \eta)$ be characteristic polynomials of orders m, m_k, v_j, respectively, $k = 1, 2, \cdots, n_1$, $j = 1, 2, \cdots, n_2$, with coefficients defined on the same closed set Ξ of $|\xi| = 1$. At each point $\xi \in \Xi$ let $P_\xi(\eta)$ be the greatest common divisor of the $P_k(\xi, \eta)$ normalized by having its leading coefficient equal to one. We take $P_\xi^+(\eta)$ to be that factor of $P_\xi(\eta)$ containing only the roots of $P_\xi(\eta)$ with non-negative imaginary parts and having them with the same multiplicity as $P_\xi(\eta)$. Thus $P_\xi^+(\eta)$ has only roots with non-negative imaginary parts and $P_\xi(\eta)/P_\xi^+(\eta)$ has only roots with negative imaginary parts. Denote the normal order of $P_\xi^+(\eta)$ by $m^+(\xi)$.

If the maximum normal order of the $P_k(\xi, \eta)$ is m_0, we shall say that Ξ has property A relative to the $P_k(\xi, \eta)$ if for at least one of them, say $P_1(\xi, \eta)$, the coefficient of η^{m_0} does not vanish anywhere in Ξ and the roots $\lambda_t(\xi)$ are single valued there.

THEOREM 3.1. *There is a constant C such that*

$$(3.1) \quad \int |(Ru)^\sim|^2 d\eta$$
$$\leq C\left\{ \sum_k |\xi|^{2(m-m_k)} \int |(P_k u)^\sim|^2 d\eta + \sum_j |\xi|^{2(m-v_j)-1} |(Q_j u)^\wedge|^2 \right\}$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$ and ξ such that $\xi/|\xi| \in \Xi$ if

(a) *there is a constant K_0 such that*

$$(3.2) \quad |R(\xi, \eta)|^2 \leq K_0 \sum_k |\xi|^{2(m-m_k)} |P_k(\xi, \eta)|^2$$

for all $\xi \in \Xi$ and real η;

(b) Ξ *has property A relative to the $P_k(\xi, \eta)$; and*

(c) *for each $\xi \in \Xi$ the polynomials $Q_j(\xi, \eta)$ cover the $P_k(\xi, \eta)$. By this we mean that there are at least $m^+(\xi)$ of the $Q_j(\xi, \eta)$ which are of normal order less than m_0 and are linearly independent modulo $P_\xi^+(\eta)$.*

The proof of Theorem 3.1 will be accomplished by means of the following lemmas.

LEMMA 3.1. *If the t_j are real and the E_j and F_j complex, then*

$$(\sum t_j |E_j|^2)(\sum t_k |F_k|^2) - |\sum t_j \bar{E}_j F_j|^2$$
$$= \frac{1}{2} \sum t_j t_k |E_j F_k - E_k F_j|^2.$$

Proof. Direct verification.

LEMMA 3.2.

$$(\sum |P_k|^2 - \varepsilon |R|^2)(\sum |(P_k u)^\sim|^2 - \varepsilon |(Ru)^\sim|^2) - |\sum \bar{P}_k (P_k u)^\sim - \varepsilon \bar{R}(Ru)^\sim|^2$$
$$= \frac{1}{2} \sum |P_k S_j - P_j S_k|^2 - \frac{1}{2} \varepsilon \sum |P_k T - R S_k|^2,$$

where

$$S_k = P_k(\xi, Lu), \quad T = R(\xi, Lu).$$

Proof. Set $t_0 = -\varepsilon$, $E_0 = R$, $F_0 = (Ru)^\sim$, $t_k = 1$, $E_k = P_k$, $F_k = (P_k u)^\sim$, $k = 1, 2, \cdots, n_1$, in Lemma 3.1.

COROLLARY 3.1.

$$(\sum |P_k|^2)(\sum |(P_k u)^\sim|^2) - |\sum \bar{P}_k (P_k u)^\sim|^2$$
$$= \frac{1}{2} \sum |P_k S_j - P_j S_k|^2.$$

LEMMA 3.3. *Let $H(\xi, \eta)$ be a characteristic polynomial and λ any complex number. If $V = H(\xi, Lu)$, then*

$$|\sum \bar{P}_k (P_k u)^\sim|^2 \geqq 2 \sum |P_k|^2 \operatorname{Re}\lambda(Hu)^\sim + 2\operatorname{Re}\lambda \sum \bar{P}_k (HS_k - VP_k) - |\lambda H|^2.$$

Proof.

$$|\sum \bar{P}_k (P_k u)^\sim|^2 = |\sum \bar{P}_k (P_k u)^\sim - \overline{\lambda H}|^2 + 2\operatorname{Re}\lambda H \sum \bar{P}_k (P_k u)^\sim - |\lambda H|^2$$
$$\geqq 2(\sum |P_k|^2) \operatorname{Re} \lambda H \tilde{u} + 2\operatorname{Re}\lambda H \sum \bar{P}_k S_k - |\lambda H|^2$$
$$= 2(\sum |P_k|^2) \operatorname{Re} \lambda(Hu)^\sim + 2\operatorname{Re}\lambda \sum \bar{P}_k (HS_k - VP_k) - |\lambda H|^2.$$

LEMMA 3.4. *For $u \in C_0^\infty(G \cup \partial_1 G)$*

$$\int \tilde{u} d\eta = -\pi i \hat{u},$$

where the integral is taken in the Cauchy principal value sense.

The proof of Lemma 3.4 was given in [12].

THEOREM 3.2. *For* $u \in C_0^\infty(G \cup \partial_1 G)$

$$\Sigma \int |P_k u|^2 d\eta \geqq \left[\int \frac{|H|^2}{\Sigma |P_k|^2} d\eta \right]^{-1} \left| \int \frac{\Sigma \bar{P}_k (HS_k - P_k V)}{\Sigma |P_k|^2} d\eta - \pi i (Hu)^\smallfrown \right|^2.$$

Proof. Set

$$J_0 = \Sigma |P_k|^2.$$

Then by Lemma 3.3 for any complex λ

$$J_0^{-1} \left| \Sigma \bar{P}_k (Pu)^\sim \right|^2 \geqq 2 \operatorname{Re} \lambda \{ J_0^{-1} \Sigma \bar{P}_k (HS_k - P_k V) + (Hu)^\sim \} - J_0^{-1} |\lambda H|^2.$$

Integrating, we have by Lemma 3.4

$$\int J_0^{-1} \left| \Sigma \bar{P}_k (P_k u)^\sim \right|^2 d\eta \geqq 2 \operatorname{Re} \lambda \left\{ \int J_0^{-1} \Sigma \bar{P}_k (HS_k - P_k V) d\eta - \pi i (Hu)^\smallfrown \right\}$$

$$- |\lambda|^2 \int J_0^{-1} |H|^2 d\eta.$$

Take

$$\bar{\lambda} = \left[\int \frac{|H|^2}{J_0} d\eta \right]^{-1} \left\{ \int \frac{\Sigma \bar{P}_k (HS_k - P_k V)}{J_0} d\eta - \pi i (Hu)^\smallfrown \right\}.$$

Thus

$$\int J_0^{-1} \left| \Sigma \bar{P}_k (P_k u)^\sim \right|^2 d\eta \geqq \left[\int \frac{|H|^2}{J_0} d\eta \right]^{-1} \left| \int \frac{\Sigma \bar{P}_k (HS_k - P_k V)}{J_0} d\eta - \pi i (Hu)^\smallfrown \right|^2.$$

Since

$$\Sigma \left| (P_k u)^\sim \right|^2 \geqq J_0^{-1} \left| \Sigma \bar{P}_k (P_k u)^\sim \right|^2$$

by Corollary 3.1, the result follows.

LEMMA 3.5. *Let*

$$P(\xi, \eta) = \sum_{s=0}^{p_1} a_s(\xi) \eta^s; \quad H(\xi, \eta) = \sum_{s=0}^{h_1} b_s(\xi) \eta^s$$

be characteristic polynomials of normal orders p *and* h, *respectively, with* $p \geqq h$. *If* $S = P(\xi, Lu)$ *and* $V = H(\xi, Lu)$, *then* $HS - PV$ *is of degree* $p - 1$ *in* η *with leading coefficient equal to* $a_p(Hu)^\smallfrown - b_p(Pu)^\smallfrown$.

Proof. Direct verification.

LEMMA 3.6. *Let* $M_1(z)$ *be a polynomial of degree* m *with leading coefficient* a. *Assume that there is a polynomial* $M_2(z)$ *of degree* $m - 1$ *with leading coefficient* b *and such that* $M_1(z)/M_2(z)$ *does not vanish for* $\operatorname{Im} z \geqq 0$. *Then*

$$\int_{-\infty}^{\infty} \frac{M_2(\tau)}{M_1(\tau)} d\tau = -\pi i \frac{a}{b},$$

where the integral is taken in the Cauchy principal value sense.

Proof. Contour integration.

LEMMA 3.7. *If the polynomials $\{Q_j(z)\}$ are linearly independent modulo a polynomial $P(z)$ and the polynomials $\{N_t(z)\}$ are linearly independent, then the set $\{Q_j(z), P(z)N_t(z)\}$ is linearly independent.*

Proof. Obvious.

We are now ready for the

Proof of Theorem 3.1. Set $m' = \max m_k$. From the statement of the theorem we see that we may assume that R and the P_k are each of order m' and that each Q_j is of order $m' - 1$. Next put

$$J_\varepsilon = \sum |P_k|^2 - \varepsilon |R|^2 = J_0 - \varepsilon |R|^2.$$

By (3.2)(⁵) if $0 < \varepsilon < \frac{1}{2} K_0^{-1}$,

$$2J_\varepsilon = J_0 + (J_0 - K_0^{-1}|R|^2) + (K_0^{-1} - 2\varepsilon)|R|^2 \geqq J_0.$$

Thus by Lemma 3.2

$$(3.3) \quad \int \left(\sum |(P_k u)^\sim|^2 - \varepsilon |(Ru)^\sim|^2 \right) d\eta \geqq -\frac{\varepsilon}{2} \int J_\varepsilon^{-1} \sum |P_k T - RS_k|^2 d\eta$$

$$\geqq -\varepsilon \int J_0^{-1} \sum |P_k T - RS_k|^2 d\eta.$$

We note that the last integral in (3.3) exists since $PT_k - RS_k$ is of degree less than m_0 in η (Lemma 3.5).

Let $H_1(\xi, \eta), H_2(\xi, \eta), \cdots, H_q(\xi, \eta)$ be the set of all characteristic polynomials of the form

$$|\xi|^{m'-\sigma-l-1} \eta^\sigma \prod_{t=1}^{l} (\eta - \lambda_t(\xi)), \quad 0 \leqq l < m_0, \quad 0 \leqq \sigma < m_0 - l,$$

where the $\lambda_t(\xi)$ are chosen from the roots of $P_1(\xi, \eta)$(⁶). Thus the characteristic polynomials $H_i(\xi, \eta)$ are of order $m' - 1$ and normal order less than m_0. By Theorem 3.2,

$$(3.4) \quad \sum_{s=1}^{q} \alpha_s(\xi) \left| \int J_0^{-1} \sum_k \bar{P}_k(H_s S_k - P_k V_s) d\eta - \pi i (H_s u)^\wedge \right|^2$$

$$\leqq q \int \sum |(P_k u)^\sim|^2 d\eta,$$

where

$$\alpha_s(\xi) = \left[\int \frac{|H_s|^2}{J_0} d\eta \right]^{-1}$$

_____

(⁵) We note that by homogeneity, (3.2) holds for all real ξ and η such that $\xi / |\xi| \in \Xi$.

(⁶) Recall that we assume that the coefficient of η^{m_0} in $P_1(\xi, \eta)$ does not vanish anywhere in Ξ and that the $\lambda_t(\xi)$ are continuous in Ξ.

and

$$V_s = H_s(\xi, Lu).$$

Clearly, $\alpha_s(\xi)$ is continuous in Ξ and homogeneous in ξ of degree one. Combining (3.3) and (3.4) we have, in view of Corollary 3.1, that there is a constant C such that

(3.5)
$$\sum \alpha_s(\xi) \left| \int J_0^{-1} \sum \bar{P}_k (H_s S_k - P_k V_s) d\eta - \pi i (H_s u)^\wedge \right|^2$$
$$+ \frac{1}{2} \int J_0^{-1} \sum |P_k S_j - P_j S_k|^2 d\eta + \sum |\xi| |Q_j u|^2$$
$$- \varepsilon \int J_0^{-1} \sum |P_k T - R S_k|^2 d\eta$$
$$\leq \int \left\{ C \sum |(P_k u)^\sim|^2 - \varepsilon |Ru|^2 \right\} d\eta + \sum |\xi| |(Q_j u)^\wedge|^2.$$

We now show that the left-hand side of (3.5) is non-negative for ε sufficiently small. Writing it in the form

$$I_1 + I_2 + I_3 - \varepsilon I_4$$

we note that each I_t is of the form

$$\sum_{i,j=0}^{m_0-1} M_{ij}(\xi) \omega_i \bar{\omega}_j,$$

where each $M_{ij}(\xi)$ is continuous in Ξ and homogeneous in ξ of degree $2m_0 + 1$. It therefore suffices to prove there is an $\varepsilon > 0$ such that

(3.6)
$$I_1 + I_2 + I_3 \geqq \varepsilon I_4$$

on the compact set $\mathscr{S}$ of all ξ, ω such that $\xi \in \Xi$ and $\sum_{i=0}^{m_0-1} |\omega_i|^2 = 1$. This will certainly be the case if

$$I_1 + I_2 + I_3$$

does not vanish on $\mathscr{S}$.

The only way I_2 can vanish for a particular ξ, ω is if

$$\frac{P_k}{P_j} \equiv \frac{S_k}{S_j}$$

identically in η, for each j, k. Since the degree of S_k in η is one less than that of P_k, this means that there is a polynomial $S(\eta)$ of degree one less than $P_\xi(\eta)$ such that

$$\frac{P_k}{S_k} \equiv \frac{P_\xi}{S}$$

identically in η for each k. In this case

(3.7)
$$\frac{\sum\limits_{k} \bar{P}_k(H_s S_k - P_k V_s)}{J_0} = \frac{H_s S - P_\xi V_s}{P_\xi}.$$

If

(3.8)
$$P_k(\xi, \eta) = \sum_{s=0}^{m'} a_{ks}(\xi)\eta^s,$$

then we see by Lemma 3.5 that the numerator on the left-hand side of (3.7) is a polynomial in η of degree $2m_0 - 1$ with leading coefficient equal to

$$(H_s u)^{\wedge} \sum |a_{km_0}(\xi)|^2,$$

while the denominator is of degree $2m_0$ and has leading coefficient equal to

$$\sum_k |a_{km_0}(\xi)|^2 \neq 0.$$

Since the leading coefficient of P_ξ is one, (3.7) tells us that $H_s S - P_\xi V_s$ is of degree one less than P_ξ and has leading coefficient equal to $(H_s u)^{\wedge}$. Moreover, among the H_s are the polynomials of the form

$$N_t(\xi, \eta) = |\xi|^{m'-t-m^+(\xi)-1} \eta^t P_\xi^+(\eta), \qquad 0 \leq t < m_0 - m^+(\xi).$$

For such polynomials Lemma 3.6 tells us that

$$\int \frac{H_s S - P_\xi V_s}{P_\xi} d\eta = -\pi i (H_s u)^{\wedge}.$$

Hence, when $I_2 = 0$,

$$\sum |\xi| |(N_t u)^{\wedge}|^2 \leq \text{const.} I_1.$$

Thus the only way $I_1 + I_2 + I_3$ can vanish is if

(3.9) $$(N_t u)^{\wedge} = 0, \qquad 0 \leq t < m_0 - m^+(\xi),$$

(3.10) $$(Q_j u)^{\wedge} = 0, \qquad 1 \leq j \leq m^+(\xi),$$

where we assume that the first $m^+(\xi)$ of the Q_j are linearly independent modulo P_ξ. This, of course, can be accomplished by renumbering them if necessary. Since the N_t are linearly independent *multiples* of P_ξ^+ and the Q_j are linearly independent *modulo* P_ξ^+, the set $\{N_t, Q_j\}$ is linearly independent (Lemma 3.7). Since there are m_0 of them and they are of degree *less* than m_0, the only way (3.9) and (3.10) can hold is if $\omega_0 = \omega_1 = \cdots = \omega_{m_0-1} = 0$. But then ξ, ω is not in $\mathscr{S}$. Hence $I_1 + I_2 + I_3$ is positive in $\mathscr{S}$ and the proof is complete.

4. The proof of Theorem 2.1. We now return to the partial differential operators R, P_k, Q_j. The corresponding characteristic polynomials are not only polynomials in η but also in the components of ξ.

We now assume that the hypotheses (a)–(c) of Theorem 2.1 are satisfied and prove that (2.5) holds for all $u \in C_0^\infty(G \cup \partial_1 G)$ and real r. By hypothesis (a) it is easily seen that the sphere $|\xi| = 1$ is the union of a finite number of closed sets Ξ_i, each of which has property A relative to the $P_k(\xi, \eta)$. In fact, let k be such that

$$\sum_{|\mu| + m_0 = m_k} a_{k\mu m_0} \xi^\mu \not\equiv 0$$

and let V be the set of those ξ on $|\xi| = 1$ such that

$$\sum_{|\mu| + m_0 = m_k} a_{k\mu m_0} \xi^\mu = 0.$$

Clearly V is compact. Moreover, hypothesis (a) tells us that for each $\xi \in V$ there is a q such that

$$(4.1) \qquad\qquad \sum_{|\mu| + m_0 = m_q} a_{q\mu m_0} \xi^\mu \neq 0$$

and hence (4.1) holds in some neighborhood of ξ. A finite covering of V by such neighborhoods gives a suitable decomposition of $|\xi| = 1$.

Next we note that (2.6) implies (3.2) for each Ξ_i. Moreover, if $P(\xi, \eta)$ is the characteristic polynomial of a constant coefficient operator P, it is easily checked by integration by parts that the Fourier transform of Pu with respect to all n variables is precisely the operator $(Pu)^\sim$ of §3. Likewise, the tangential Fourier transform of Pu is $(Pu)^\wedge$ as defined there (cf. [12]).

Hence, by Theorem 3.1, there are constants C_i such that

$$\int |(Ru)^\sim|^2 d\eta \leq C_i \left\{ \sum_k |\xi|^{2(m-m_k)} \int |(P_k u)^\sim|^2 d\eta + \sum_j |\xi|^{2(m-v_j)-1} |(Q_j u)^\wedge|^2 \right\}$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$ and ξ such that $\xi/|\xi| \in \Xi_i$. Summing over i, we have that

$$\int |(Ru)^\sim|^2 d\eta \leq C \left\{ \sum_k |\xi|^{2(m-m_k)} \int |(P_k u)^\sim|^2 d\eta + \sum_j |\xi|^{2(m-v_j)-1} |(Q_j u)^\wedge|^2 \right\}$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$ and $\xi \neq 0$. We now multiply both sides by $|\xi|^{2r}$ and integrate with respect to ξ. This gives (2.5) and the proof is complete.

5. **Bounded domains.** We now assume that G is contained in the strip $0 \leq x_n \leq d$. We shall say that Ξ has property B relative to a characteristic polynomial $P(\xi, \eta)$ if each root $\lambda(\xi)$ of $P(\xi, \eta)$ is single valued and continuous on Ξ and satisfies either

(a) Im $\lambda(\xi) > 0$ at all interior points of Ξ, or

(b) Im $\lambda(\xi) \leq 0$ in Ξ.

If Ξ has property B relative to P, we let $P^{s, t}(\xi, \eta)$ stand for the generic polynomial of the form

$$\frac{P(\xi, \eta)}{\prod\limits_{q=1}^{s} (\eta - \alpha_q(\xi)) \prod\limits_{q=1}^{t} (\eta - \beta_q(\xi))},$$

where the $\alpha_q(\xi)$ satisfy (a) and the $\beta_q(\xi)$ satisfy (b). If either s or t is negative, we set $P^{s,\,t}(\xi, \eta) \equiv 0$. We retain the notation of §3 and state

THEOREM 5.1. *For any set of non-negative integers ρ, $\{l_k\}$ satisfying $\rho \leqq \max l_k$, there is a constant C such that*

$$
(5.1) \quad
\begin{aligned}
\int |(Ru)^{\sim}|^2 d\eta \leqq C \Bigg\{ &\sum_k |\xi|^{2(m-m_k+l_k)} \int |(P_k u)^{\sim}|^2 d\eta \\
&+ \sum_j |\xi|^{2(m-\nu_j)-1} \Bigg(1 + \sum_{s=0}^{\rho} |\xi|^{2s+1} \Bigg) |(Q_j u)^{\wedge}|^2 \Bigg\}
\end{aligned}
$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$ and ξ such that $\xi/|\xi| \in \Xi$, if

(a) Ξ *has property* B *relative to each $P_k(\xi, \eta)$ and property* A *relative to the polynomials $P_k^{0,\,l_k}(\xi, \eta)$;*

(b) *there are non-negative integers $\rho_k \leqq \min(\rho + 1, l_k)$ such that*

$$
(5.2) \quad |R(\xi, \eta)|^2 \leqq K \sum_k \sum_{s=0}^{\rho_k} |\xi|^{2(m-m_k+l_k)} |P_k^{s,\,l_k-s}(\xi, \eta)|^2
$$

for all real η and $\xi \in \Xi$, where summation is taken over all polynomials of the form $P_k^{s,\,l_k-s}(\xi, \eta)$;

(c) *the $Q_j(\xi, \eta)$ cover the polynomials $P_k^{0,\,\sigma_k}(\xi, \eta)$, where*

$$\sigma_k = \min(l_k - \rho_k, \, l_k - \rho).$$

We shall prove Theorem 5.1 at the end of this section. We first derive some consequences. Setting ρ and the ρ_k equal to zero in Theorem 5.1 we have

COROLLARY 5.1. *For any non-negative integers l_k there is a constant C such that*

$$
\begin{aligned}
&\int |(Ru)^{\sim}|^2 d\eta \\
&\leqq C \Bigg\{ \sum_k |\xi|^{2(m-m_k+l_k)} \int |(P_k u)^{\sim}|^2 d\eta + \sum_j (1 + |\xi|) |\xi|^{2(m-\nu_j)-1} |(Q_j u)^{\wedge}|^2 \Bigg\}
\end{aligned}
$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$ and ξ such that $\xi/|\xi| \in \Xi$, if

(a) Ξ *has property* B *relative to each $P_k(\xi, \eta)$ and property* A *relative to the polynomials $P_k^{0,\,l_k}(\xi, \eta)$;*

(b) *there is a constant K such that*

$$
(5.3) \quad |R(\xi, \eta)|^2 \leqq K \sum_k |\xi|^{2(m-m_k+l_k)} |P_k^{0\,\,l_k}(\xi, \eta)|^2
$$

for real η and $\xi \in \Xi$;

(c) *the $Q_j(\xi, \eta)$ cover the polynomials $P_k^{0,\,l_k}(\xi, \eta)$.*

REMARK 5.1. If we take all the l_k to be zero in Corollary 5.1 we obtain

$$\int |Ru|^2 d\eta$$
$$\leq C\left\{\sum_k |\xi|^{2(m-m_k)} \int |P_k u|^2 d\eta + \sum_j (1+|\xi|)|\xi|^{2(m-\nu_j)-1}|Q_j u|^2\right\},$$

which is slightly weaker than inequality (3.1).

COROLLARY 5.2. *Inequality* (5.1) *holds for non-negative integers* ρ *and* l_k *such that* $\rho \leq \max l_k$, *if*

(a) Ξ *has property* B *relative to each* $P_k(\xi,\eta)$ *and property* A *relative to the* $P_k^{0,l_k}(\xi,\eta)$;

(b) *there is a constant* K *such that*

$$(5.4) \qquad |R(\xi,\eta)|^2 \leq K \sum_k |\xi|^{2(m-m_k+l_k)} \sum_{t=0}^{\rho+1} |P_k^{t,l_k-t}(\xi,\eta)|^2$$

for real η *and* $\xi \in \Xi$;

(c) *the* $Q_j(\xi,\eta)$ *cover the polynomials* $P_k^{0,l_k-\rho}(\xi,\eta)$.

Proof. Take $\rho_k = \min(\rho+1, l_k)$ in Theorem 5.1. Clearly (5.4) implies (5.2). Moreover, since $l_k - \rho \geq \sigma_k$, if the Q_j cover the P_k^{0,σ_k}, they surely cover the $P_k^{0,l_k-\rho}$.

COROLLARY 5.3. *Assume that* $m_1 = m + 1$. *Then*

$$\int |(Ru)^\sim|^2 d\eta \leq C\left\{\int |(P_1 u)^\sim|^2 d\eta + \sum_j (1+|\xi|)|\xi|^{2(m-\nu_j)-1}|(Q_j u)^\wedge|^2\right\}$$

if Ξ *has properties* A *and* B *relative to* $P_1(\xi,\eta)$,

$$|R(\xi,\eta)|^2 \leq K \sum (|P_1^{0,1}(\xi,\eta)|^2 + |P_1^{1,0}(\xi,\eta)|^2),$$

and the Q_j *cover the* $P_1^{0,1}(\xi,\eta)$.

In certain applications it is convenient to take some of the $P_k(\xi,\eta)$ to be the same. For instance we have

THEOREM 5.2. *For any set of non-negative integers* $\rho, \{l_k\}$ *with* $\rho \leq \max l_k$, *there is a constant* C *such that*

$$(5.5) \qquad \int |(Ru)^\sim|^2 d\eta \leq C\left\{\sum_k \sum_{s=0}^{l_k} |\xi|^{2(m-m_k+s)} \int |(P_k u)^\sim|^2 d\eta\right.$$
$$\left. + \sum_j |\xi|^{2(m-\nu_j)-1}(1+\sum_{s=0}^{\rho}|\xi|^{2s+1})|(Q_j u)^\wedge|^2\right\}$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$ *and* ξ *satisfying* $\xi/|\xi| \in \Xi$, *if*

(a) Ξ *has property* A *relative to the* $P_k(\xi,\eta)$ *and property* B *relative to each* $P_k(\xi,\eta)$;

(b) *there are non-negative integers* $\rho_k \leqq \min(\rho + 1, l_k)$ *such that*

$$(5.6) \qquad |R(\xi,\eta)|^2 \leqq K \sum_k \sum_{s=0}^{\rho_k} \sum_{t=0}^{l_k-s} |\xi|^{2(m-m_k+s+t)} |P_k^{s,t}(\xi,\eta)|^2$$

for real η *and* $\xi \in \Xi$;

(c) *the* $Q_j(\xi,\eta)$ *cover the* $P_k^{0,\sigma_k}(\xi,\eta)$, $\sigma_k = \min(l_k - \rho_k, l_k - \rho)$.

Proof. Set $P_{ks}(\xi,\eta) = P_k(\xi,\eta)$, $l_{ks} = s$, $\rho_{ks} = \rho_k$, $s = 0, 1, \cdots, l_k$, and apply Theorem 5.1. One sees immediately that hypotheses (a) and (b) are satisfied. As for (c), we note that the Q_j cover the polynomials $P_k^{0,\sigma_{kk}}$ where

$$\sigma_{ks} = \min(l_{ks} - \rho_{ks}, l_{ks} - \rho).$$

Hence they cover the larger set $P_k^{0,\sigma_{ks}}$ (cf. Lemma 5.4 below), and the proof is complete.

The proof of Theorem 5.1 will be based upon the following lemmas.

LEMMA 5.1. *For* $t \geqq 0$ *and* $s \geqq 1$ *there is a constant* C *depending on* d *such that for any characteristic polynomial* $P(\xi,\eta)$

$$(5.7) \qquad \int |(P^{0,t}u)^\sim|^2 d\eta \leqq C \int |(Pu)^\sim|^2 d\eta,$$

$$(5.8) \qquad |(P^{0,t}u)^\wedge|^2 \leqq C \int |(Pu)^\sim|^2 d\eta,$$

$$(5.9) \qquad \int |(P^{s,t}u)^\sim|^2 d\eta \leqq C \left\{ \int |(Pu)^\sim|^2 d\eta + \sum_{q=1}^{s} |(P^{q,t}u)^\wedge|^2 \right\}$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$ *and* ξ *such that* $\xi/|\xi| \in \Xi$.

Proof. The lemma follows from Parseval's identity and repeated applications of the inequalities

$$(5.10) \qquad |g(t)| \leqq C \int_0^d |(D - i\lambda)g(\tau)| d\tau, \qquad\qquad \mathrm{Im}\lambda \leqq 0,$$

$$(5.11) \qquad |g(t)| \leqq C \left(\int_0^d |(D - i\lambda)g(\tau)| d\tau + |g(0)| \right), \qquad \mathrm{Im}\lambda > 0,$$

where D denotes differentiation with respect to τ. The inequalities (5.10) and (5.11) hold for differentiable functions $g(t)$ in $0 \leqq t \leqq d$ which vanish for $t = d$. They are well known and easily proved (cf., e.g., [14]).

LEMMA 5.2. *If*

$$P(\xi,\eta) = \eta - \beta(\xi)$$

is a characteristic polynomial, then

$$(5.12) \qquad |\mathrm{Im}\beta|^2 \int |\tilde{u}|^2 d\eta \leqq \int |(Pu)^\sim|^2 d\eta + 2\pi \mathrm{Im}\beta |\hat{u}|^2$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$. *G need not be bounded for* (5.12) *to hold*.

Proof. We have

$$|(Pu)^\sim|^2 = |(\eta - \beta)\tilde{u} + \hat{u}|^2 = |\eta - \beta|^2 |\tilde{u}|^2 + 2\operatorname{Re}(\eta - \beta)\tilde{u}\hat{u}^- + |\hat{u}|^2$$

$$= [\eta^2 - 2\operatorname{Re}\beta\eta + (\operatorname{Re}\beta)^2 + (\operatorname{Im}\beta)^2]|\tilde{u}|^2 + 2\operatorname{Re}(\eta - \beta)\tilde{u}\hat{u}^- + |\hat{u}|^2$$

$$= (\eta - \operatorname{Re}\beta)^2|\tilde{u}|^2 + 2\operatorname{Re}(\eta - \operatorname{Re}\beta)\tilde{u}\hat{u}^- + |\hat{u}|^2 + (\operatorname{Im}\beta)^2|\tilde{u}|^2$$

$$\qquad + 2\operatorname{Re}(-i\operatorname{Im}\beta)\tilde{u}\hat{u}^- = |(\eta - \operatorname{Re}\beta)\tilde{u} + \hat{u}|^2 + (\operatorname{Im}\beta)^2|\tilde{u}|^2$$

$$\qquad + 2\operatorname{Re}(-i\operatorname{Im}\beta)\tilde{u}\hat{u}^-.$$

Hence

$$\int |(Pu)^\sim|^2 d\eta \geqq (\operatorname{Im}\beta)^2 \int |\tilde{u}|^2 d\eta + 2\operatorname{Re}(-i\operatorname{Im}\beta)\hat{u}^- \int \tilde{u}\, d\eta$$

$$= (\operatorname{Im}\beta)^2 \int |\tilde{u}|^2 d\eta - 2\pi\operatorname{Im}\beta|\hat{u}|^2$$

by Lemma 3.4, and the proof is complete.

LEMMA 5.3. *Assume that $\beta(\xi)$ is a root of the characteristic polynomial $P(\xi, \eta)$ of multiplicity $> s$ and such that $\operatorname{Im}\beta(\xi) \leqq 0$ in Ξ. Set*

$$P_s^\beta(\xi, \eta) = \frac{P(\xi, \eta)}{(\eta - \beta(\xi))^s}.$$

Then for $t \leqq s$

(5.13) $$|\operatorname{Im}\beta|^{2(t-1)}|(P_s^\beta u)^\wedge|^2 \leqq C \int |(Pu)^\sim|^2 d\eta.$$

Proof. By repeated use of (5.12) we have, since $\operatorname{Im}\beta \leqq 0$,

$$|\operatorname{Im}\beta|^{2(t-1)} \int |(P_{t-1}^\beta u)^\sim|^2 d\eta \leqq \int |(Pu)^\sim|^2 d\eta.$$

We then apply (5.8) to complete the proof.

LEMMA 5.4. *If the Q_j cover a subset of the P_k, then they cover the whole set.*

Proof. Trivial (cf. [12, p. 55]).

Proof of Theorem 5.1. Since each $\sigma_k \leqq l_k - \rho_k$, the Q_j cover the polynomials $P_k^{0, l_k}(\xi, \eta)$ and hence, by Lemma 5.4, the larger set $P_k^{s, l_k - s}(\xi, \eta)$, $0 \leqq s \leqq \rho_k$. Applying Theorem 3.1, we see, by hypothesis (b), that

$$\int |(Ru)^\sim|^2 d\eta$$

(5.14)

$$\leqq C \left\{ \sum_k \sum_{s=0}^{\rho_k} |\xi|^{2(m-m_k+l_k)} \int |(P_k^{s, l_k-s}u)^\sim|^2 d\eta + \sum_j |\xi|^{2(m-\nu_j)-1}|Q_j u|^2 \right\}.$$

Now by Lemma 5.1, for any particular κ

$$(5.15) \qquad \int |(P_\kappa^{0,\ lk-s}u)^\sim|^2 d\eta \leqq C \int |(P_\kappa u)^\sim|^2 d\eta$$

and

$$(5.16) \qquad \int |(P_\kappa^{s,\ lk-s}u)^\sim|^2 d\eta \leqq C\left\{ \int |(P_\kappa u)^\sim|^2 d\eta + \sum_{q=1}^{s} |(P_\kappa^{q,lk-s}u)^\wedge|^2 \right\}$$

for $s \geqq 1$. We are going to show that

$$|\xi|^{2(m-m_\kappa+l_\kappa)} |(P_\kappa^{q,\ lk-s}u)^\wedge|^2$$

$$\begin{aligned}(5.17) \qquad &\leqq C\Bigg\{ \sum_k |\xi|^{2(m-m_k+l_k)} |(P_k^{0,\ lk-s+q}u)^\wedge|^2 \\ &+ \sum_{k,h} \sum_{\rho=lk-s+q+1}^{\rho_{kh}} |\xi|^{2(m-m_k+l_k)} |\mathrm{Im}\beta_{kh}|^{2(\rho+s-lk-q)} |(P_{\kappa\rho}^{\beta_{kh}}u)^\wedge|^2 \\ &+ \sum_j |\xi|^{2(m+s-\nu_j-q)} |(Q_j u)^\wedge|^2 \Bigg\} \end{aligned}$$

for each κ, s, q such that $1 \leqq q \leqq s \leqq \rho_\kappa$ where the $\beta_{kh}(\xi)$ are the roots of $P_k(\xi,\eta)$ which have nonpositive imaginary parts in Ξ and are of multiplicity $\rho_{kh} \geqq l_k - s + q + 1$. Moreover, it is to be understood that the term containing the $\beta_{kh}(\xi)$ in (5.17) is to appear only when $l_k - s + q + 1 \geqq 2$.

Once (5.17) is proved, the theorem then follows from (5.14–5.17) and the fact that each term on the right-hand side of (5.17) can be estimated in terms of the right-hand side of (5.1). This is clear for the first term by (5.7) (recall that $P^{0,\ lk-s+q} \equiv 0$ if $l_k - s + q < 0$). For the middle term we employ Lemma 5.3, making use of the fact that this term comes into play only when $l_k - s + q \geqq 1$. As for the third term, it is obviously contained in the second term on the right-hand side of (5.1).

It therefore remains only to prove (5.17). Each term in (5.17) is of the form

$$\sum M_{ih}(\xi)\omega_i\bar\omega_h ,$$

where the $M_{ih}(\xi)$ are continuous in ξ and homogeneous of degree $m + s - q + 2$. Following the proof of Theorem 3.1 we see that we must show that the vanishing of the right-hand side of (5.17) implies that $\omega_0 = \omega_1 = \cdots = \omega_{m_0^s,\ q-1} = 0$, when $m_0^{s,\ q}$ is the greatest normal order of the $P_k^{0,\ lk-s+q}$. But this is indeed the case. For let $P_\xi(\eta)$ be the greatest common divisor of the $P_k^{0,\ lk-s+q}$ and let $P_\xi^+(\eta)$ be its factor containing its roots with non-negative imaginary parts. Denote the normal order of $P_\xi^+(\eta)$ by $m^+(\xi)$. Since $l_k - s + q \geqq \sigma_k$, we see by hypothesis (c) that there are $m^+(\xi)$ of the Q_j which are linearly independent modulo $P_\xi^+(\eta)$. Moreover, if at ξ there are no common roots of the $P_k^{0,\ lk-s+q}$ with negative imaginary parts and having multiplicities greater than one, then there are $m_0^{s,\ q} - m^+(\xi)$ polynomials among them which are linearly independent. Other-

wise we adjoin to them the polynomials $P_{k\rho}^{\beta_{kh}}$ corresponding to such roots. It should be noticed that the factors $\operatorname{Im} \beta_{kh}$ accompanying the $P_{k\rho}^{\beta_{kh}}$ in (5.17) do not vanish for such roots. Thus we see that there are $m_0^{s,\,q} - m^+ (\xi)$ polynomials among the $P_k^{0,\,l_k-s+q}$ and $P_{k\rho}^{\beta_{kh}}$ which are linearly independent. Since the latter all contain $P_\xi^+(\eta)$ as a factor, we have among the $P_k^{0,\,l_k-s+q}$, $P_{ks}^{\beta_{kh}}$, and Q_j, $m_0^{s,\,q}$ polynomials which are linearly independent (Lemma 3.7). Since they are all of normal order less than $m_0^{s,\,q}$ the simultaneous vanishing of the

$$(P_k^{0,\,l_k-s+q}u)\hat{\ }, \quad (P_{k\rho}^{\beta_{kh}}u)\hat{\ }, \quad (Q_ju)\hat{\ }$$

mplies that $\omega_0 = \omega_1 = \cdots = \omega_{m_0^s,\,q-1} = 0$ and the proof is complete.

6. Further results and the remaining proofs. An immediate consequence of Theorem 5.1 is

THEOREM 6.1. *For any set of non-negative integers* $\rho, \{l_k\}$ *with* $\rho \leqq \max l_k$, *there is a constant C such that*

$$(6.1) \qquad |Ru|_{r-m} \leqq C \left\{ \sum_k |P_ku|_{r-m_k+l_k} + \sum_j \sum_{s=0}^{\rho+1} \langle Q_ju \rangle_{r-\nu_j+s-1} \right\}$$

for all real r and $u \in C_0^\infty(G \cup \partial_1 G)$, *if the sphere* $|\xi| = 1$ *is the union of a finite number of closed sets* Ξ_i *on each of which hypotheses* (a)–(c) *of Theorem 5.1 hold.*

The proof is obvious. By Theorem 5.1, inequality (5.1) holds for each Ξ_i. Summing over all the Ξ_i, multiplying by $|\xi|^{2(r-m)}$, employing the trivial inequality $2 \leqq |\xi|^{-1} + |\xi|$, and integrating with respect to ξ, we obtain (6.1).

REMARK 6.1. Statements similar to Theorem 6.1 hold in connection with Corollaries 5.1–5.3 and Theorem 6.2. For the latter we obtain the inequality

$$(6.2) \qquad |Ru|_{r-m} \leqq C \left\{ \sum_k \sum_{s=0}^{l_k} |P_ku|_{r-m_k+s} + \sum_j \sum_{s=0}^{\rho+1} \langle Q_ju \rangle_{r-\nu_j+s-1} \right\}$$

when $|\xi| = 1$ can be broken up into a finite number of closed sets on each of which hypotheses (a)–(c) of Theorem 5.2 hold.

When G is bounded and r is sufficiently large we can simplify (6.2) by employing an inequality due to Schwartz [13].

LEMMA 6.1. *Let D be a bounded domain in* E^{n-1} *and h a non-negative real number. Then there is a constant C depending only upon h and the size of D such that*

$$(6.3) \qquad \int |\xi|^{2h}|\hat{\phi}|^2 d\xi \leqq C \int_{|\xi|>1} |\xi|^{2h}|\hat{\phi}|^2 d\xi$$

for all functions $\phi(x')$ *with compact support in D.*

For the proof of Lemma 6.1 we refer to Schwartz [13, Theorem 6.1]. Employing it we have

THEOREM 6.2. *Assume that* $|\xi| = 1$ *is the union of a finite number of closed sets on each of which hypotheses* (a)–(c) *of Theorem 5.2 hold. Then if G is bounded, there is a constant C such that*

$$(6.4) \qquad |Ru|_{r-m} \leq C\left\{ \sum_k |P_k u|_{r-m_k+l_k} + \sum_j \langle Q_j u \rangle_{r-v_j+\rho} \right\}$$

for all $u \in C_0^\infty(G \cup \partial_1 G)$ *and real* $r \geq \max_{j,k}(m_k, v_j + 1)$.

Proof. We apply Lemma 6.1 to inequality (6.2). By assumption, the powers of $|\xi|$ in the integrals on the right-hand side of (6.2) are non-negative. Hence by Lemma 6.1 they are majorized by a constant times the same integrals over $|\xi| > 1$. But on this set $|\xi|^{2(r-m_k+s)} \leq |\xi|^{2(r-m_k+l_k)}$, and the proof is complete.

We now give the remaining proofs of the theorem of §2.

Proof of Theorem 2.2. We first note that $|\xi| = 1$ can be subdivided into a finite number of closed sets Ξ_i each of which has property B relative to each $P_k(\xi, \eta)$. Next, by hypothesis (a) each Ξ_i can be subdivided into a finite number of closed sets Ξ_{ih} each of which has property A relative to the $P_k(\xi, \eta)$. Thus by Theorem 6.2, we need only show that hypotheses (b) and (c) of Theorem 5.2 are fulfilled in each Ξ_{ih}.

It is easily seen that (2.8) implies (5.6). In fact

$$(6.5) \qquad \left| \frac{\partial^s P_k(\xi, \eta)}{\partial \eta^s} \right| \leq \text{const.} \sum_{t=0}^{s} |P_k^{t,\,s-t}(\xi, \eta)|^2,$$

and if $t \leq s \leq \rho_k \leq l_k$, then $s - t \leq l_k - t$. Thus the terms on the right-hand side of (6.5) are among the terms on the right-hand side of (5.6).

Finally, we note that for each ξ and k, $P_k^{(\sigma_k)}(\xi, \eta)$ is the greatest common divisor for all polynomials of the form $P_k^{0,\,\sigma_k}(\xi, \eta)$. Hence the Q_j cover the $P_k^{0,\,\sigma_k}(\xi, \eta)$ if and only if they cover the $P_k^{(\sigma_k)}(\xi, \eta)$. Thus the proof is complete.

Proof of Theorem 2.3. As before, we can prove the theorem by showing that hypotheses (b) and (c) of Theorem 5.2 are satisfied for any closed set Ξ of $|\xi| = 1$ which has property A relative to the $P_k(\xi, \eta)$ and property B relative to each $P_k(\xi, \eta)$. Thus we may assume that

$$\sum a_{1\mu m_0} \xi^\mu \neq 0$$

for $\xi \in \Xi$. If

$$\theta = \min_{\xi \in \Xi} \left| \sum a_{1\mu m_0} \xi^\mu \right|,$$

then $\theta > 0$ and

$$|P_1(\xi, \eta)| \geq \theta \eta^{m_0} - |E(\xi, \eta)|$$

for $\xi \in \Xi$, where $E(\xi, \eta)$ is a polynomial in η of degree less than m_0 with coefficients depending on ξ which are uniformly bounded for $\xi \in \Xi$. Thus by (a′) there are contants K and M such that

(6.6)
$$\frac{|R(\xi, \eta)|}{|P_1(\xi, \eta)|} \leq K$$

for $\xi \in \Xi$ and $|\eta| > M$. Moreover, by (b') the polynomials $P_k^{0,t}(\xi, \eta)$,

$$0 \leq t \leq \gamma = \min(\rho + 1, \min l_k)$$

cannot all vanish for any real ξ, η such that $|\xi| = 1$. Taking each $\rho_k = \gamma$, we see that there is a constant K such that (5.6) holds on the compact set $\xi \in \Xi$, $-M \leq \eta \leq M$. Combining this with (6.6) we see that (5.6) holds for all real ξ, η such that $\xi \in \Xi$.

Finally, since $\sigma_k \leq l_k - \rho$, we see that the Q_j cover the $P_k^{0, l_k - \rho}(\xi, \eta)$ and hence the $P_k^{(l_k - s)}(\xi, \eta)$. This completes the proof.

Proof of Theorem 2.4. Here we apply Corollary 5.1 (cf. Remark 6.1). We note that in two dimensions the whole set $|\xi| = 1$ (i.e., $\xi = \pm 1$) has properties A and B relative to any and all polynomials. Moreover,

$$\left| \frac{\partial^{s+t} P_k(\xi, \eta)}{\partial \xi^s \partial \eta^t} \right|^2 \leq \text{const.} \sum |P_k^{0, s+t}(\xi, \eta)|^2$$

and hence (2.10) implies (5.3). Finally, we observe, as before, that $P^{(h)}(\xi, \eta)$ is the greatest common divisor of the polynomials $P^{0,h}(\xi, \eta)$. Thus hypothesis (c) of Corollary 5.1 is also satisfied and the proof is complete.

The proof of Theorem 2.5 relies upon the following fact

LEMMA 6.2. *If $R(\xi, \eta)$ and the $P_k(\xi, \eta)$ are homogeneous, each $m_k \geq m$, and $R(\xi, \eta)$ is weaker than the $P_k(\xi, \eta)$, then there is a constant C such that*

(6.7)
$$|R(\xi, \eta)|^2 \leq C \sum_{s+t=m_k-m} \left| \frac{\partial^{s+t} P_k(\xi, \eta)}{\partial \xi^s \partial \eta^t} \right|^2$$

for all real ξ, η.

Proof. Let m_0 be the maximum normal order of the $P_k(\xi, \eta)$ and m_0' the normal order of $R(\xi, \eta)$. Then by (2.11), $m_0 \geq m_0'$. Assume that $P_1(\xi, \eta)$ is of normal order m_0. Then there is a derivative of $P_1(\xi, \eta)$ of order $m_1 - m$ which is of degree $\geq m_0'$ in η. If $m_0 \geq m$, it is

$$\frac{\partial^{m_1 - m} P_1(\xi, \eta)}{\partial \xi^{m_1 - m_0} \partial \eta^{m_0 - m}}$$

and if $m_0 < m$, it is

$$\frac{\partial^{m_1 - m} P_1(\xi, \eta)}{\partial \xi^{m_1 - m}}.$$

Thus as any event, there are constants C and M such that (6.7) holds for $\xi = \pm 1$ and $|\eta| > M$.

Now the only way (6.7) can fail to hold on the compact set $\xi = \pm 1$, $|\eta| \leqq M$ is when all the derivatives on the right-hand side have a real root α with multiplicity $\geqq h \geqq 1$ and α is either not a root of $R(\xi, \eta)$ or a root with multiplicity less than h. Assume this to be the case. Since the $P_k(\xi, \eta)$ are homogeneous, Euler's formula shows us that α is a root of all derivatives of $P_k(\xi, \eta)$ of order $\leqq m_k - m$. Hence α is a root of $P_k(\xi, \eta)$ with multiplicity at least $m_k - m + h$. Thus any derivative of $P_k(\xi, \eta)$ of order $p \leqq m_k - m + h$ has α as a root with multiplicity $m_k - m + h - p$. The degree in η of such a derivative is, of course, $m_k - p$.

Now set $\xi_N = N$, $\eta_N = \alpha N + 1$. Then $\eta_N - \alpha\xi = 1$, while $|\eta_N - \beta\xi_N| \to \infty$ as $N \to \infty$ if $\beta \neq \alpha$. By the considerations above, we see that

$$\frac{\partial^{s+t} P_k(\xi_N, \eta_N)}{\partial^s \xi \partial^t \eta} = O(N^{m-h})$$

for all s and t. Since α is not a root of $R(\xi, \eta)$ with multiplicity $\geqq h$, we know that $|R(\xi_N, \eta_N)|/N^{m-h} \to \infty$, and hence (2.11) cannot hold. Thus our assumption is false and (6.7) must hold on the set $\xi = \pm 1$ and $|\eta| \leqq M$. Hence it holds for $|\xi| = 1$ and all real η. By homogeneity, it holds for all real ξ, η, and the proof is complete.

Proof of Theorem 2.5. We set $\tau = r - m$, $l_k = m_k - m$ in Theorem 2.4. By Lemma 6.2, (6.7) holds. Moreover the right-hand side of (6.7) is majorized by

$$\text{const.} \sum |P_k^{0, \, m_k - m}(\xi, \eta)|^2.$$

The theorem now follows from Corollary 5.1 and Remark 6.1.

Added in Proof. The methods of the present paper can be adapted to give corresponding inequalities for operators which are not necessarily homogeneous. These inequalities permit one to obtain regularity up to the boundary for solutions of hypoelliptic boundary problems. Details will appear elsewhere.

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